CRYSTALLISATION AND SCHUR-WEYL DUALITY

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Abstract. The crystal analog of Schur-Weyl Duality is not yet fully understood. Schur-Weyl Duality provides an up-to-scalar unique isomorphism that decomposes $(\mathbb{C}^n)^{\otimes d}$ into a space with either $\mathrm{GL}_n(\mathbb{C})$-representations or S_d-representations. For any $\mathrm{GL}_n(\mathbb{C})$-representation, there is an associated crystal. Under this crystallisation, Schur-Weyl duality's isomorphism coincides with the Robinson–Schensted–Knuth Correspondence (RSK). Both the domain and co-domain of RSK have canonical Cactus Group (C_d) actions. The C_d-action on the co-domain is described directly using the partial Schützenberger involution on Tableau , and the C_d-action on the domain uses the general commutator of crystals on the specific crystal $B_{\omega_1}^{\otimes d}$. It is yet to be proven that the RSK map is equivariant with respect to these two actions. This research will prove that RSK is C_d-equivariant.

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1. Introduction

1.1. Representation Theory.

Representation theory aims to *represent* an algebraic structure using linear transformations of a vector space. In this perspective, an element of the algebra will be a matrix, and algebraic operations are performed through matrix operations. As linear transformation and matrices are well understood through Linear Algebra, representation theory can simplify abstract algebraic structures.

Key to representation theory is an *action*. For a representation of a group or algebra G, a left G-action on a set V is an operator $(\ \cdot\ ) : G \times V \to V$, with the most important property being that $g_1 \cdot (g_2 \cdot v) = (g_1 g_2) \cdot v$. There is also an analogous right G-action on V which respects right multiplication of the group elements.

1.2. Cactus Group.

Definition 1.2.1. The *Cactus Group* C_d is generated by elements $\mathfrak{q}_{[i,j]}$ for $1 \le i < j \le d$. The elements of C_d follow the relations:

- $\mathfrak{q}_{[i,j]}^2 = 1$
- $\mathfrak{q}_{[i,j]}\mathfrak{q}_{[k,l]} = \mathfrak{q}_{[k,l]}\mathfrak{q}_{[i,j]}$ if $j < k$
- $\mathfrak{q}_{[i,j]}\mathfrak{q}_{[k,l]}\mathfrak{q}_{[i,j]} = \mathfrak{q}_{[i+j-l,i+j-k]}$ if $i \le k < l \le j$

In the context of this research, the importance of C_d does not intrinsically come from its definition, but rather from the natural actions that it induces on certain sets. Specifically, C_d generates actions on the domain and co-domain of the Robinson–Schensted–Knuth Correspondence.

1.3. **The Robinson–Schensted–Knuth Correspondence (RSK).**

The Robinson–Schensted–Knuth correspondence is a bijection between the set of Words of length d in the alphabet $[n]$ and the set of pairs of Semi-Standard and Standard Young Tableau of size d and length at most n.

RSK was first proposed by Knuth in his book *The Art of Computer Programming: Vol 3: Sorting and Searching* [Knu98]. This was a generalisation of the Robinson–Schensted correspondence, which provided a similar bijection, with a domain of permutations (rather than RSK's domain of Words).

This correspondence was part of Knuth's exploration of sorting algorithms. Many in-place sorting algorithms can be viewed as a specific composition of permutations, which rearrange elements to sort them. By understanding permutations, Knuth was able to enrich his exploration of sorting algorithms, as well as other abstract data types including non-cyclic directed graphs.

1.4. **Schur-Weyl Duality.**

Schur-Weyl Duality was developed by Schur [Boe48, Sch01] and popularised by Weyl in his book *The Classical Groups: Their Invariants and Representations* [Wey66]. This book was one of the foundational texts of group and representation theory.

Schur-Weyl Duality provides a duality between irreducible representations of the symmetric group S_d and irreducible representations of the general linear group $\mathrm{GL}_n(\mathbb{C})$.

Schur-Weyl Duality is given by the up-to-scalar unique isomorphism:

$$(\mathbb{C}^n)^{\otimes d} \cong \bigoplus_{\lambda} V_\lambda \otimes S^\lambda$$

V_λ and S^λ are irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ and S_d respectively, and λ is the index of partitions of d with length at most n. This isomorphism allows for dual decompositions of $(\mathbb{C}^n)^{\otimes d}$ into spaces with either $\mathrm{GL}_n(\mathbb{C})$-representations, or S_d-representations.

1.5. **Crystallisation.**

Each V_λ has an associated crystal, which allows for a canonical crystallisation of Schur-Weyl duality. The formal definition of 'crystallisation' is not required for this research. When Schur-Weyl Duality is crystallised, the underlying map coincides with RSK.

Under the view of Shur-Weyl Duality as a decomposition of $\mathrm{GL}_n(\mathbb{C})$-representations, this crystallisation is well understood. However from the perspective of S_d-representations, the crystallisation is not fully developed, which is something that this research aims to address. This research aims to show that the canonical C_d-actions are equivariant for the S_d-decomposition's crystallisation.

1.6. **Cactus Group Actions.**

Both the domain and co-domain of RSK have established left C_d-actions. Standard Young Tableau have a C_d-action described by the *partial Schützenberger involution* [LY23]. The set of Words can be viewed as a crystal, which naturally endues a C_d-action [HK05]

1.7. **Aim.**

This research aims to prove the C_d-equivariance of RSK, i.e. the C_d-action and the RSK map commute. Formally this equivariance follows if for all words $b \in [n]^d$ and $\mathfrak{q}_{[i,j]} \in C_d$

$$\mathrm{RSK}\left(\mathfrak{q}_{[i,j]} \cdot b\right) = \mathfrak{q}_{[i,j]} \cdot \mathrm{RSK}\,(b)$$

On the left-hand side, C_d is acting on words, whilst on the right-hand side C_d is acting on pairs of Young Tableau.

This will be achieved by defining and exploring these two actions, and leveraging established RSK identities and crystal theorems.

2. Representation Theory

2.1. **Representations.**

Definition 2.1.1. A *representation* of a group G on a vector space V is a group homomorphism from G to $\mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is the general linear group of V. That is, $\rho : G \to \mathrm{GL}(V)$ is a G-representation of V if for all $g_1, g_2 \in G$

$$\rho(g_1 \cdot g_2) = \rho(g_1) \cdot \rho(g_2)$$

Such a function ρ allows for an arbitrary group operator to be 'redefined' via the composition of linear transformations, which are well understood through matrices and Linear Algebra.

2.2. **Actions.**

Definition 2.2.1. For G a group or algebra, a *left G-action* on a set V is a map $\alpha : G \times V \to V$ such that

- $\alpha(1, v) = v$ for 1 the identity element of G and for all $v \in V$.
- $\alpha(g_1, \alpha(g_2, v)) = \alpha(g_1 g_2, v)$ for all $v \in V$ and $g_1, g_2 \in G$.

If such a map exists, V is called a *G-set*.

The map α is often written as an operator $\alpha(g, v) =: g \cdot v$.

2.3. **Equivariance.**

Definition 2.3.1. A map $f : X \to Y$ is *G-equivariant* if X and Y are G-sets, and for all $x \in X$ and $g \in G$,

$$f(g \cdot x) = g \cdot f(x)$$

3. Words

3.1. Definitions.

Definition 3.1.1. An *alphabet* $\mathcal{A}$ of size n is a non-empty, finite set of letters a_j such that $\mathcal{A} = \{a_1, \ldots, a_n\}$.

In this research, the main alphabet used will be $[n] := \{1, \ldots, n\}$.

Definition 3.1.2. A *word* w of length d in the alphabet $\mathcal{A}$ is an ordered multi-set of letters $w = (w_1, \ldots w_d)$ such that $w_i \in \mathcal{A}$.

$[n]^d$ is the set of words of length d in the alphabet $[n]$.

4. Young Tableau

4.1. Definitions.

Definition 4.1.1. A *partition* λ of n, denoted $\lambda \vdash n$, is a sequence of non-negative integers $(\lambda_1, \ldots, \lambda_i)$ such that $\lambda_1 \geq \cdots \geq \lambda_i$ and $\sum\limits_{j \in [i]} \lambda_j = n$. The length of a partition is $l(\lambda) := i$.

Definition 4.1.2. A *Young diagram* of shape λ and size n is a collection of left-justified boxes such that $\lambda \vdash n$ and the j-th row of the diagram has λ_j boxes.

Definition 4.1.3. A *Young Tableau* (also shortened to just *Tableau*) of shape λ in the alphabet $\mathcal{A}$ is obtained by filling in each box of a Young diagram of the same shape with a letter from the alphabet $\mathcal{A}$. The letter that is in a box is called the box's *label*. When a Tableau is denoted as $T(\lambda)$, it is implied that its shape is λ.

Definition 4.1.4. A Tableau is *semi-standard* if its entries are **weakly** increasing across rows and strictly increasing down columns. Take the set of Semi-Standard Young Tableau of shape $\lambda \vdash n$ as $SSYT(\lambda)$.

Definition 4.1.5. A Tableau is *standard* if its entries are **strictly** increasing across rows and strictly increasing down columns. Take the set of Standard Young Tableau of shape $\lambda \vdash n$ as $SYT(\lambda)$.

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array}$$

Figure 1. An example standard Young Tableau of shape $(4, 2, 1)$.

Definition 4.1.6. A *skew Young Tableau* of shape λ/μ is a Tableau of shape λ with any boxes that would also be in $T(\mu)$ removed.

		1	3
	2		
4			

FIGURE 2. An example skew Young Tableau of shape $(4, 2, 1)/(2, 1)$.

Definition 4.1.7. The *restriction* of a Tableau T (denoted $T|_{[a,b]}$) is a new, possibly skewed Tableau that disregards any box that has its label outside the interval $[a, b]$ (where $a \leq b$).

4.2. **Operations on Tableau.**

Definition 4.2.1. *Jeu de Taquin* is a map $jdt : SSYT(\lambda/\mu) \to SSYT(\lambda')$ that transforms a semi-standard skew Tableau into a semi-standard Tableau of the same size. Below is the algorithmic description of how $jdt(T)$ is computed.

(1) Take a removed box in μ and call it *movable*.

(2) Choose a movable box of T and consider its east-adjacent box $\boxed{i}$ and south-adjacent box $\boxed{j}$.

   (a) If a movable box has no east or south-adjacent box, then do nothing.

   (b) If the movable box has only one east or south-adjacent box, swap the movable box with this adjacent box (noting that a box stays movable after it is moved).

   (c) If $i < j$ then swap the movable box with $\boxed{i}$.

   (d) Otherwise, swap the movable box with $\boxed{j}$.

(3) Repeat (2) until all movable boxes have no east or south-adjacent box.

(4) Disregard all movable boxes.

Proposition 4.2.2 ([LY23])**.** *jdt is well defined, i.e. the map is independent of the choice of order in moving each box.*

The jdt process of the example skew Tableau in Figure 2 is computed in Figure 3, where movable boxes are light grey and the most recently moved box is dark grey.

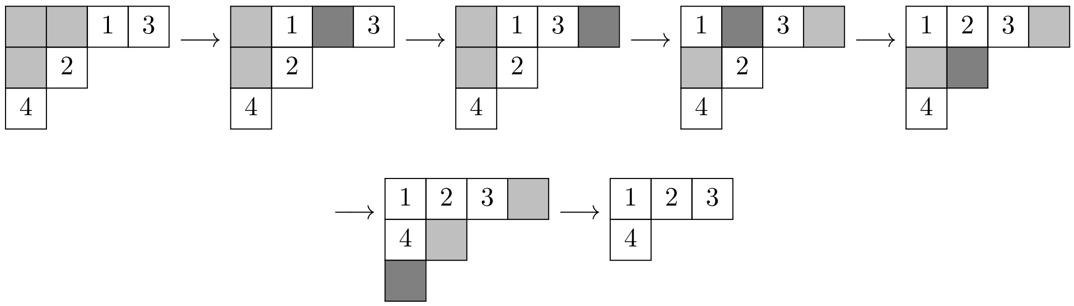

FIGURE 3. Jeu de Taquin process on the standard skew Tableau of Figure 2.

Definition 4.2.3. The *promotion* map $\partial : SSYT(\lambda) \to SSYT(\lambda)$ in the alphabet $[n]$ is defined as follows.

(1) Turn every box labelled 1 into a movable box.

(2) Apply jdt to these movable boxes.

(3) Reduce each non-movable box's label by 1.

(4) Relabel the movable boxes to n.

This operation can be extended to ∂_k which acts on T as $\partial\left(T\mid_{[1,k]}\right)$ in the alphabet $[k]$ whilst leaving $T\mid_{[k+1,n]}$ unchanged.

Definition 4.2.4 ([LY23])**.** The *Schützenberger involution* of T in the alphabet $[n]$ is the map $\xi : SYT(\lambda) \to SYT(\lambda)$ defined by

$$\xi(T) := \partial_1 \circ \partial_2 \circ \ldots \circ \partial_n(T)$$

This operation is also known as *evacuation* in the literature.

Proposition 4.2.5 ([Sch72])**.** *ξ is an involution.*

Definition 4.2.6 ([LY23])**.** Take $\lambda \vdash n$. For $1 < k \leq n$ the *partial Schützenberger involution* $\xi_{[1,k]} : SYT(\lambda) \to SYT(\lambda)$ is defined to be the Schützenberger involution acting on $T\mid_{[1,k]}$ in the alphabet $[k]$ and leaving $T\mid_{[k+1,n]}$ unchanged. This is equivalent to $\xi_{[1,k]} := \partial_1 \circ \partial_2 \circ \ldots \circ \partial_k$.

This can be extended once more to $\xi_{[a,b]} := \xi_{[1,b]} \circ \xi_{[1,b-a+1]} \circ \xi_{[1,b]}$ for all $1 \leq a < b \leq n$.

Proposition 4.2.7 ([KB95])**.** *$\xi_{[a,b]}$ acting on SYT follows the Cactus Group relations of 1.2.1.*

Definition 4.2.8. Consider $\lambda \vdash d$ and $l(\lambda) \leq n$.

The left C_d-action on $SYT(\lambda)$ in the alphabet $[d]$ is defined as $\mathfrak{q}_{[a,b]} \cdot T := \xi_{[a,b]}(T)$.

5. Crystals

5.1. Definitions.

Definition 5.1.1 ([HK05])**.** Let $\mathfrak{g}$ be a complex reducible Lie Algebra, Λ its weight lattice, Λ_+ its set of dominant weights, I the set of vertices in its Dynkin diagram, $\{\alpha_i \mid i \in I\}$ its simple roots and $\{\alpha_i^\vee \mid i \in I\}$ its simple co-roots.

A $\mathfrak{g}$-crystal is a finite set B with associated maps

$$\begin{aligned} \mathrm{wt} &: B \to \Lambda \\ \varepsilon_i &: B \to \mathbb{Z} \\ \phi_i &: B \to \mathbb{Z} \\ e_i &: B \to B \sqcup \{0\} \\ f_i &: B \to B \sqcup \{0\} \end{aligned}$$

such that for all $i \in I$ and $b \in B$, the following relationships hold:

- $\phi_i(b) - \varepsilon_i(b) = \langle \mathrm{wt}(b), \alpha_i^\vee \rangle$.
- $\varepsilon_i(b) = \max\{n \mid e_i^n \cdot b \neq 0\}$ with e_i^n being n successive applications of e_i.
- $\phi_i(b) = \max\{n \mid f_i^n \cdot b \neq 0\}$ with f_i^n being n successive applications of f_i.
- If $e_i \cdot b \neq 0$ then $\mathrm{wt}(e_i \cdot b) = \mathrm{wt}(b) + \alpha_i$.
- If $f_i \cdot b \neq 0$ then $\mathrm{wt}(f_i \cdot b) = \mathrm{wt}(b) - \alpha_i$.
- For all $b, b' \in B$, then $b' = e_i \cdot b$ if and only if $b = f_i \cdot b'$.

The maps f_i and e_i are known as the *Kashiwara operators*.

For the main result of this research, the first fundamental weight crystal B_{ω_1} will be considered which has

$$\begin{aligned} \mathfrak{g} &= \mathfrak{gl}_n \\ B &= [n] \\ I &= \{1, \dots, n-1\} \\ \Lambda &= \bigoplus_{i \in I} \mathbb{Z}\varpi_i \\ \Lambda_+ &= \bigoplus_{i \in I} \mathbb{N}\varpi_i \\ \alpha_i &= (\underbrace{0, \dots, 0}_{i-1}, 1, -1, \underbrace{0, \dots, 0}_{n-i-1}) \end{aligned}$$

where $\varpi_i = (\underbrace{1, \dots, 1}_{i}, \underbrace{0, \dots, 0}_{n-i})$.

Crystals can be viewed less abstractly using a crystal diagram. If $b \in B$ are placed as the vertices of a directed graph, the edges of the crystal graph can be computed using the f_i operators. An i-edge will go from vertex b to b' if $f_i \cdot b = b'$.

Under this crystal diagram, the maps f_i move along i-edges, and e_i move against i-edges. This is why the application of f_i and e_i are sometimes described as a *move* in this research.

When restricting a crystal graph to just its i-edges, the map $\phi_i(b)$ counts how many f_i moves that can be applied until the end of b's connected component is reached, travelling with the edge direction. Similarly, $\varepsilon_i(b)$ counts how many e_i moves that can be applied to reach the start of b's connected component, travelling against the edge direction.

Definition 5.1.2 ([HK05])**.** A crystal B_λ is a *highest weight crystal of highest weight* λ if there exists a $b \in B_\lambda$ such that $\mathrm{wt}(b) = \lambda$, $e_i \cdot b = 0$ for all $i \in I$ and B_λ is generated by a finite number of f_i moves applied to b.

b is known as the *highest weight element* of the crystal. There is an analogous *lowest weight element*, which is sent to 0 by all f_i, and B_λ is generated by e_i moves.

A general crystal can have many highest and lowest weight elements, with at most one for each of it's connected sub-crystal.

Definition 5.1.3 ([Tor16])**.** For A, B crystals, $\varphi : A \to B$ is a *crystal isomorphism* if φ is a bijective map of sets $\varphi : A \sqcup \{0\} \to B \sqcup \{0\}$ such that

(1) $\varphi(0) = 0$.

(2) For all $a \in A$, then $\mathrm{wt}(\varphi(a)) = \mathrm{wt}(a)$, $\varepsilon_i(\varphi(a)) = \varepsilon_i(a)$ and $\phi_i(\varphi(a)) = \phi_i(a)$.

(3) For all $a \in A$ with $f_i(a) \neq 0$, then $f_i(\varphi(a)) = \varphi(f_i(a))$.

(4) For all $a \in A$ with $e_i(a) \neq 0$, then $e_i(\varphi(a)) = \varphi(e_i(a))$.

Under this definition of a crystal isomorphism, φ preserves all crystal structure that has been defined thus far. For example, if a is a highest weight element of weight λ, then $\varphi(a)$ is also a highest weight element of weight λ in the crystal B.

5.2. **The Set of Words as a $\mathfrak{gl}_n$-Crystal.**

The *first fundamental crystal of* $\mathfrak{gl}_n$ denoted B_{ω_1} is a $\mathfrak{gl}_n$-crystal which will be used to represent the set of words $[n]^d$. The crystal diagram of B_{ω_1} is below.

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \cdots\cdots\cdots n$$

FIGURE 4. Cyrtsal diagram of B_{ω_1}.

Definition 5.2.1 ([HK05])**.** The *tensor product* $A \otimes B$ of two $\mathfrak{g}$-crystals A and B is defined with the following properties. The set of vertices becomes $A \times B$, with $\mathrm{wt}(a, b) = \mathrm{wt}(a) + \mathrm{wt}(b)$.

The e_i and f_i maps follow:

$$e_i \cdot (a,b) = \begin{cases} (e_i \cdot a, b), & \text{if } \varepsilon_i(a) > \phi_i(b) \\ (a, e_i \cdot b), & \text{otherwise} \end{cases}$$
$$f_i \cdot (a,b) = \begin{cases} (f_i \cdot a, b), & \text{if } \varepsilon_i(a) \geq \phi_i(b) \\ (a, f_i \cdot b), & \text{otherwise} \end{cases}$$

The tensor product is crucial to the crystal representation of words. By taking the d-fold tensor product of B_{ω_1}, denoted $B_{\omega_1}^{\otimes d}$, the underlying vertex set becomes $[n]^d$, which is the domain of RSK.

Proposition 5.2.2 ([HK05])**.** *The tensor product of crystals is associative.*

The map

$$\begin{aligned} \alpha_{A,B,C} : (A \otimes B) \otimes C &\to A \otimes (B \otimes C) \\ ((a,b),c) &\mapsto (a,(b,c)) \end{aligned}$$

is a crystal isomorphism, so parentheses in multiple tensor products can be dropped [HK05].

5.3. The Commutator.

The *commutator* is a crystal isomorphism which is a C_d-action on crystals. To define this commutator, the natural automorphism of crystal graphs $\xi_B : B \to B$ will be used.

Definition 5.3.1 ([HK05])**.** Take B_λ a connected sub-crystal of a $\mathfrak{g}$-crystal B. The map $\xi_{B_\lambda} : B_\lambda \to B_\lambda$ is the unique map that follows:

$$\begin{aligned} e_i \cdot \xi(b) &= \xi\left(f_{n-i} \cdot b\right) \\ f_i \cdot \xi(b) &= \xi\left(e_{n-i} \cdot b\right) \\ \mathrm{wt}(\xi(b)) &= w_0 \cdot \mathrm{wt}(b) \end{aligned}$$

From this definition, ξ_{B_λ} exchanges the highest and lowest weight elements of B_λ, and the image of all vertices can be computed by expressing them as a finite number of f_i moves applied to the highest weight element, or a finite number of e_i moves applied to the lowest weight element.

ξ_{B_λ} can be extended to the whole crystal B by applying ξ_{B_λ} to each connected sub-crystal B_λ of B. This extended operator acting on the whole crystal B is denoted ξ.

Definition 5.3.2 ([HK05])**.** The commutator $\sigma_{A,B}$ is defined as

$$\begin{aligned} \sigma_{A,B} : A \otimes B &\to B \otimes A \\ (a,b) &\mapsto \xi\left(\xi(b), \xi(a)\right) \end{aligned}$$

This map is an isomorphism of crystals [HK05]. This commutator can be extended to more complex tensor products whilst remaining an isomorphism.

Definition 5.3.3 ([HK05])**.** The commutator $\sigma_{[1,k]}$ is defined as

$$\begin{aligned}\sigma_{[1,k]}:\bigotimes_{i\in[d]} B_i &\to B_k\otimes\ldots\otimes B_1\otimes B_{k+1}\otimes\ldots\otimes B_d\\ (b_1,\ldots,b_d)&\mapsto\left(\xi\left(\xi\left(b_k\right),\ldots,\xi\left(b_1\right)\right),b_{k+1}\ldots,b_d\right)\end{aligned}$$

and $\sigma_{[a,b]}:=\sigma_{[1,b]}\circ\sigma_{[1,b-a+1]}\circ\sigma_{[1,b]}$ for all $1\leq a<b\leq d$.

Proposition 5.3.4 ([HK05])**.** *The operator $\sigma_{[a,b]}$ follows the Cactus Group relations of 1.2.1.*

Definition 5.3.5. The left C_d-action on words $b\in B_{\omega_1}^{\otimes d}$ is defined as $\mathfrak{q}_{[a,b]}\cdot b:=\sigma_{[a,b]}(b)$.

6. The Robinson–Schensted–Knuth Correspondence

6.1. **Schensted Insertion.**

Definition 6.1.1 ([SF99])**.** *Schensted Insertion* is an operation on semi-standard Young Tableau which inserts a new element a into the Tableau T, whilst keeping it semi-standard.

Take $T\in SSYT(\lambda)$ in the alphabet $[n]$, with entry $T(i,j)$ in the i-th row and j-th column, and $\lambda=(\lambda_1,\ldots,\lambda_{l(\lambda)})$ a partition of d.

Define a new semi-standard Young Tableau $T\leftarrow a$, named the insertion of a into T. The Tableau $T\leftarrow a$ will have one new box and one new label a, not necessarily in this new box. $T\leftarrow a$ can be computed using the following algorithmic description.

If $a\geq T(1,\lambda_1)$, then create a new box at $T(1,\lambda_1+1)$, place a inside and terminate the algorithm.

Otherwise, take j as the left-most entry in the first row that is strictly greater than a. Replace this entry j with a. It is said that a *bumps* j. Then insert j using the same process as above, but now into the second row.

If there exists a left-most entry $h>j$ in the second row, replace h with j and bump h, which is subsequently inserted into the third row. If no such entry exists (i.e. $j\geq T(2,\lambda_2)$) then a new box is created at $T(2,\lambda_2+1)$ and j is placed into this new box, terminating the algorithm.

If the end of the Tableau is reached without creating a new box, one is created at position $T(l(\lambda)+1,1)$ and the last bumped entry is placed into the new box.

Example 6.1.2. Take T as the example in 4.1.5. $T\leftarrow 1$ is computed in Figure 5. The circled number is that which was most recently bumped.

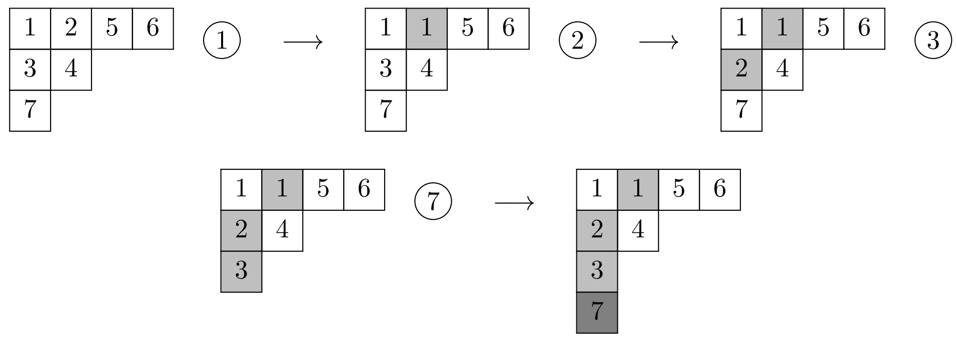

FIGURE 5. Computation of $T \leftarrow 1$ for the standard Tableau of Figure 1.

Definition 6.1.3. If μ is the shape of $T \leftarrow a$, then μ/λ is a single box and is known as the *new location* of the insertion of a into T (the dark grey box in Figure 5).

Proposition 6.1.4 ([SF99])**.** *If the new location is known, then the Schensted insert is reversible. That is, T can be recovered from $T \leftarrow a$ by backtracking the Schensted insertion.*

6.2. **The Robinson–Schensted–Knuth Correspondence.**

Definition 6.2.1 ([Knu98])**.** The Robinson–Schensted–Knuth correspondence (RSK) is a bijection

$$\mathrm{RSK} : [n]^d \to \bigsqcup_{\substack{\lambda \vdash d \\ l(\lambda) \leq n}} SSYT(\lambda) \times SYT(\lambda)$$

Let $b = (b_1, \dots b_d) \in [n]^d$.

To compute $\mathrm{RSK}(b) = (P, Q)$, find the *insertion Tableau* P using repeated Schensted insertions

$$P = \varnothing \leftarrow b_1 \leftarrow b_2 \leftarrow \cdots \leftarrow b_d$$

and the *recording Tableaux* Q, which has entry i in the box corresponding to the new location of the insertion of b_i into P.

$\mathrm{RSK}^{-1}(P, Q)$ can be computed by backtracking the Schensted insertions from P starting at the final new location. This final new location will be the box containing d in Q.

Definition 6.2.2. The left C_d-action on

$$(P, Q) \in \bigsqcup_{\substack{\lambda \vdash d \\ l(\lambda) \leq n}} SSYT(\lambda) \times SYT(\lambda)$$

acts as $\mathfrak{q}_{[a,b]} \cdot (P, Q) := (P, \mathfrak{q}_{[a,b]} \cdot Q)$.

This is a valid C_d action as $\mathfrak{q}_{[a,b]} \cdot Q$ is a C_d-action on $STY(\lambda)$ (4.2.8).

7. Main Result

7.1. **Main Proposition.**

Proposition 7.1.1. *The RSK correspondence is C_d-equivariant. That is, for all words $b \in [n]^d$ and $\mathfrak{q}_{[i,j]} \in C_d$*

$$RSK\left(\mathfrak{q}_{[i,j]} \cdot w\right) = \mathfrak{q}_{[i,j]} \cdot RSK(w)$$

This will be proven using some auxiliary results.

7.2. **Auxiliary Results.**

Lemma 7.2.1 ([Shi05]). *For $b \in B_{\omega_1}$, the following are true*

- 1 *is the highest weight element.*
- n *is the lowest weight element.*
- $f_i(i) = i + 1$.
- $e_i(i + 1) = i$.

Lemma 7.2.2. *For $i \in B_{\omega_1}$, then $\xi(i) = n + 1 - i$.*

Proof.

If $i = 1$, then $\xi(1) = n = n + 1 - 1$. If $i > 1$

$$\begin{aligned} \xi(i) &= \xi(f_{i-1} \cdot \ldots \cdot f_1(1)) \\ &= e_{n+1-i} \cdot \ldots \cdot e_{n-1}(\xi(1)) \\ &= e_{n+1-i} \cdot \ldots \cdot e_{n-1}(n) \\ &= n + 1 - i \end{aligned}$$

□

Lemma 7.2.3 ([Knu98]). *For a word $b = (b_1, \ldots, b_d)$ in the alphabet $[n]$ where $RSK(b) = (P, Q)$, then for $b^* := (b_d^*, \ldots, b_1^*)$, $RSK(b^*) = (\xi P, \xi Q)$, with $b_i^* = n + 1 - b_i$.*

Lemma 7.2.4. *If b is a highest weight element of $B_{\omega_1}^{\otimes d}$, then b^* is a lowest weight element of $B_{\omega_1}^{\otimes d}$.*

Proof.

Take $b \in B_{\omega_1}^{\otimes d}$ a highest weight element.

As $\sigma_{[1,d]}$ is a crystal isomorphism, then it maps a highest weight element to another highest weight element. It follows that

$$\begin{aligned}\sigma_{[1,d]}(b) &= \xi(\xi(b_d), \dots, \xi(b_1)) \\ &= \xi(n+1-b_d, \dots, n+1-b_1) \\ &= \xi(b^*) \quad \text{which is a highest weight crystal}\end{aligned}$$

As ξ maps a lowest weight crystal to a highest weight crystal, then b^* is a lowest weight element. $\square$

Lemma 7.2.5. *For a crystal isomorphism $\varphi : A \to B$ and any highest weight element $a \in A$, then $\varphi(\xi(a)) = \xi(\varphi(a))$.*

Proof.

The following diagram commutes for highest weight elements $a \in A$:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\xi"] & B \arrow[d, "\xi"] \\ A \arrow[r, "\varphi"] & B \end{tikzcd}$$

Following 'right then down', for a a highest height crystal of weight λ of A, it will be mapped to the unique highest weight element of weight λ in B under φ. Then it will be mapped to the corresponding lowest weight element in B under ξ.

Following 'down then right', for a a highest height crystal of weight λ of A, it will be mapped to the unique lowest weight element of weight in a's connected component under ξ. Then it will be mapped to the corresponding lowest weight element in B under φ.

Hence for any highest weight element $a \in A$, then $\phi(\xi(a)) = \xi(\phi(a))$. $\square$

Lemma 7.2.6 ([BS17])**.** *RSK is a crystal isomorphism between $B_{\omega_1}^{\otimes d} \to \bigsqcup_{\substack{\lambda \vdash d \\ l(\lambda) \leq n}} SSYT(\lambda) \times SYT(\lambda)$, with all crystal operations acting on the P-Tableau (when $RSK(b) = (P, Q)$).*

Lemma 7.2.7 ([LLT95])**.** *In the case $\mathfrak{g} = \mathfrak{gl}_n$, with the tableaux model for B_λ, ξ_{B_λ} coincides with the Schützenberger involution on Young Tableau.*

Theorem 7.2.8. *For b a highest weight element of $B_{\omega_1}^{\otimes d}$ and $RSK(b) = (P, Q)$, then $\mathfrak{q}_{[1,d]} \cdot b = RSK^{-1}(\mathfrak{q}_{[1,d]} \cdot (P, Q))$.*

Proof. This can be proven directly from the above lemmas.

$$\begin{aligned}\mathfrak{q}_{[1,d]} \cdot b &= \xi(\xi(b_d), \dots, \xi(b_1)) \\ &= \xi(n+1-b_d, \dots n+1-b_1) \\ &= \xi(b^*) \\ &= \mathrm{RSK}^{-1}(\mathrm{RSK}(\xi(b^*))) \\ &= \mathrm{RSK}^{-1}(\xi(\mathrm{RSK}(b^*))) \\ &= \mathrm{RSK}^{-1}(\xi(\xi P, \xi Q)) \\ &= \mathrm{RSK}^{-1}(P, \xi Q) \\ &= \mathrm{RSK}^{-1}(\mathfrak{q}_{[1,d]} \cdot (P,Q))\end{aligned}$$

□

Theorem 7.2.9. *For b a highest weight element of $B_{\omega_1}^{\otimes d}$ and RSK$(b) = (P,Q)$, then $\mathfrak{q}_{[1,i]} \cdot b =$ RSK$^{-1}(\mathfrak{q}_{[1,i]} \cdot (P,Q))$.*

Proof.

It is enough to show $\mathrm{RSK}(\mathfrak{q}_{[1,i]} \cdot b|_{[1,i]}, b|_{[i+1,d]}) = (P, \xi|_{[1,i]} Q)$, where the two arguments of RSK is shorthand for RSK being computed on the concatenation of the two words.

Let $\mathrm{RSK}(b) = (P,Q)$ and $\mathrm{RSK}(b|_{[1,i]}) = (P_i, Q_i)$.

It follows from 7.2.8 that $\mathrm{RSK}(\mathfrak{q}_{[1,i]} \cdot b|_{[1,i]}) = (P_i, \xi_{[1,i]} Q_i)$.

To compute $\mathrm{RSK}(\mathfrak{q}_{[1,i]} \cdot b|_{[1,i]}, b|_{[i+1,d]})$, inductively insert the elements $b_{i+1}, \dots, b_d$ using the definition of RSK.

As the P tableau of $\mathfrak{q}_{[1,i]} \cdot b|_{[1,i]}$ and $b|_{[1,i]}$ are the same, the new location of the insertion of b_{i+1} into this tableau will be identical. It follows that when appending b_{i+1} to either $\mathfrak{q}_{[1,i]} \cdot b|_{[1,i]}$ or $b|_{[1,i]}$, the element $i+1$ will have the same cell location in the Q tableau of $(\mathfrak{q}_{[1,i]} \cdot b|_{[1,i]}, b_{i+1})$ and $(b|_{[1,i]}, b_{i+1})$.

Inductively appending up to b_d, observe that the P tableau of $(\mathfrak{q}_{[1,i]} \cdot b|_{[1,i]}, b|_{[i+1,d]})$ and b are identical, and all cells with label greater that i in Q coincide for $(\mathfrak{q}_{[1,i]} \cdot b|_{[1,i]}, b|_{[i+1,d]})$ and b. From the definition of the partial Schützenberger involution, it follows that $\mathrm{RSK}(\mathfrak{q}_{[1,i]} \cdot b|_{[1,i]}, b|_{[i+1,d]}) = (P, \xi|_{[1,i]} Q)$ as required. □

Corollary 7.2.10. *For $b \in B_{\omega_1}^{\otimes d}$ and RSK$(b) = (P,Q)$, then $\mathfrak{q}_{[1,k]} \cdot b =$ RSK$^{-1}(\mathfrak{q}_{[1,k]} \cdot (P,Q))$.*

Proof.

This result can be proven by induction on the amount of f_i moves applied to $b^{\text{h.w.e}}$ to compute b, where $b^{\text{h.w.e}}$ is the highest weight element in b's connected component.

The base case is when $b = b^{\text{h.w.e}}$, i.e. zero f_i moves required to get from $b^{\text{h.w.e}}$ to b. It was proven in 7.2.9 that $\mathfrak{q}_{[1,i]} \cdot b = \mathrm{RSK}^{-1}(\mathfrak{q}_{[1,i]} \cdot (P,Q))$ when $b = b^{\text{h.w.e}}$.

Assume F is a composition of a finite number of f_i moves such that $b' = F(b^{\text{h.w.e}})$ and $\mathfrak{q}_{[1,i]} \cdot b' = \text{RSK}^{-1}(\mathfrak{q}_{[1,i]} \cdot (P', Q'))$.

Consider $b = f_i \cdot F(b^{\text{h.w.e}}) = f_i(b')$. As RSK is a crystal isomorphism, $(P, Q) = \text{RSK}(b) = \text{RSK}(f_i(b')) = f_i(\text{RSK}(b')) = f_i(P', Q')$. Hence

$$\begin{aligned}
\mathfrak{q}_{[1,k]} \cdot b &= \mathfrak{q}_{[1,k]} \cdot (f_i \cdot F(b^{\text{h.w.e}})) \\
&= \mathfrak{q}_{[1,k]} \cdot f_i(b') \\
&= f_i(\mathfrak{q}_{[1,k]} \cdot (b')) \\
&= f_i(\text{RSK}^{-1}(\mathfrak{q}_{[1,k]} \cdot (P', Q'))) \\
&= \text{RSK}^{-1}(f_i(\mathfrak{q}_{[1,k]} \cdot (P', Q'))) \\
&= \text{RSK}^{-1}(\mathfrak{q}_{[1,k]} \cdot f_i(P', Q')) \\
&= \text{RSK}^{-1}(\mathfrak{q}_{[1,k]} \cdot (P, Q))
\end{aligned}$$

□

7.3. **Proof of Main Theorem (7.1.1).**

Proof.

Take $b \in [n]^d$ and $\mathfrak{q}_{[a,b]} \in C_d$.

$$\begin{aligned}
\text{RSK}(\mathfrak{q}_{[a,b]} \cdot b) &= \text{RSK}(\mathfrak{q}_{[1,b]} \cdot \mathfrak{q}_{[1,b-a+1]} \cdot \mathfrak{q}_{[1,b]} \cdot b) \\
&= \mathfrak{q}_{[1,b]} \cdot \mathfrak{q}_{[1,b-a+1]} \cdot \mathfrak{q}_{[1,b]} \cdot \text{RSK}(b) \\
&= \mathfrak{q}_{[a,b]} \cdot \text{RSK}(b)
\end{aligned}$$

□

8. Conclusion

This research was able to prove the C_d-equivariance of the RSK correspondence, providing a deeper understanding of Schur-Weyl Duality's crystallisation of S_d-representations.

The formulations of the C_d-actions on the domain and co-domain of RSK follow the canonical descriptions established in the literature. The C_d-action on Tableau is described directly using the partial Schützenberger involution, and the C_d-action on words uses the general commutator of crystals on the specific crystal $B_{\omega_1}^{\otimes d}$.

Such descriptions induce certain limitations for this research. Whilst the canonical descriptions of the two actions were used, they can seem 'disjointed' from each other. The action on Tableau is directly described for any given Tableau, whilst the action on a given word requires extra information. Specifically, the whole set of Words (represented as a crystal) is needed for the action to be computed on a given element. This fundamental difference in how the two actions are described is one reason why RSK's equivariance could not be proved more directly.

It is conjectured that there is an algorithmic description of the C_d-action on words which can be computed directly for a given word, which would allow a more direct proof of RSK's C_d-equivariance which does not use crystal theory.

9. Future Directions

As part of exploratory exercises into the C_d-action on words, a description of the action independent of crystal theory and RSK was attempted to be found. Whilst some progress was made, this was unsuccessful. Such an algorithm could provide a deeper insight into the C_d-action on words.

9.1. **Knuth Relations and Knuth Equivalence.**

There are two Knuth relations.

Two words x and y differ by a Knuth relation of the first kind if for $a < b < c$, $x = (x_1, \ldots, b, a, c, \ldots, x_d)$ and $y = (x_1, \ldots, b, c, a, \ldots, x_d)$, where the b letter is in the same position in x and y.

Two words x and y differ by a Knuth relation of the second kind if for $a < b < c$, $x = (x_1, \ldots, a, c, b, \ldots, x_d)$ and $y = (x_1, \ldots, c, a, b, \ldots, x_d)$, where the b letter is in the same position in x and y.

Two words are Knuth equivalent if they differ by a finite sequence of Knuth relations.

It was conjectured that there exists a finite sequence of Knuth relations that can be algorithmically determined which describes the C_d-action on words. The reasons for this are:

(1) Two words are Knuth equivalent if and only if they have the same P tableau under the RSK correspondence [Vo12].

(2) The P Tableau of a word remains unchanged under the C_d-action on words.

It can be deduced that there must exist a finite sequence of Knuth relations that relate b and $\mathfrak{q}_{[a,b]} \cdot b$, yet the algorithmic description of which Knuth relations to use is unclear.

If one can find such an algorithm that describes this finite sequence of Knuth relations, this could produce a more direct proof of RSK's C_d-equivariance, rather than using crystal theory.

9.2. **Schur-Weyl Duality Types.**

During this research, the Schur-Weyl Duality that was crystallised to produce RSK is known as Type A Schur-Weyl Duality. However, other types of Schur-Weyl Duality can be crystallised. These crystallisation will produce new crystal isomorphism, with associated correspondence.

Such 'RSK correspondences' for other Schur-Weyl Duality Types are yet to be investigated in the literature, and may produce interesting interactions with other groups.

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